Constrained Control of a Nonlinear Two Point Boundary Value Problem, I

Dedicated to Lothar von Wolfersdorf on the occasion of his 60th birthday

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Abstract. In this paper we consider an optimal control problem for a nonlinear second order ordinary differential equation with integral constraints. A necessary optimality condition in form of the Pontryagin minimum principle is derived. The proof is based on McShane-variations of the optimal control, a thorough study of their behaviour in dependence of some defining parameters, a generalized Green formula for second order ordinary differential equations with measurable coefficients and certain tools of convex analysis.

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1. Introduction

In some previous papers [5, 7, 9, 10] we have dealt with optimal control problems for *second order* ordinary differential equations. Thereby our intention was to work and formulate the results within the framework of the given state equation and not to reformulate the original control problem into a control problem for a *system of two first order* differential equations. The latter may be (formally) possible in case of quasilinear state equation, however is in principle impossible for non-quasilinear equation, where the defining functions are not smooth enough. For such a state equation we have considered an unconstrained control problem in [10]. Here, for the same state equation we are going to investigate a constrained control problem, which will be briefly denoted by (**P**) and which reads as follows.
Find

$$\inf \mathcal{T}_0(u, y)$$

subject to

$$u \in U_{ad} = \{u \in L^\infty_m(0,1)\colon u(x) \in Q \ a.e.\ x \in (0,1)\}\,, \tag{1}$$

$$-\frac{d}{dx}a(x, y(x), y'(x)) + b(x, u(x), y(x), y'(x)) = 0\,, \quad x \in (0,1)\,, \tag{2}$$
$$y(0) = 0\,, \quad y(1) = 0\,,$$

$$\mathcal{T}_\nu(u, y) \leqslant 0\,, \quad \nu = -m_0, \ldots, -1\,, \quad \mathcal{T}_\nu(u, y) = 0\,, \quad \nu = 1, \ldots, n_0\,, \tag{3}$$

where $L^\infty_m(0,1)$ stands for the m-fold Cartesian product of $L^\infty(0,1)$ and $Q \subset \mathbb{R}^m$, $m \geqslant 1$, is a given set called control set, containing at least two elements. The functionals $\mathcal{T}_\nu$, $\nu = -m_0, \ldots, n_0$, are given by

$$\mathcal{T}_\nu(u, y) = \int_0^1 g_\nu(x, u(x), y(x), y(x), y'(x))\, dx\,. \tag{4}$$

The given real-valued functions $a = a(x, s, t)$, $b = b(x, r, s, t)$ and $g_\nu = g_\nu(x, r, s, t)$ are defined for $x \in (0,1)$, $r \in Q, s, t \in \mathbb{R}$ and satisfy certain assumptions to be specified below.

The characteristic feature of this control problem (**P**) is that the function a in the state equation (2) does not depend on the control $u \in U_{ad}$. Just because of this we are able to derive a necessary optimality condition in the form of a Pontryagin minimum principle (Theorem 1), from which we get a linearized (weak) Pontryagin minimum principle (Theorem 2) supposed the control set $Q \subset \mathbb{R}^m$ is convex. To prove these results we use McShane-variations $u_{\varepsilon c}$ ($\varepsilon > 0, c \in \mathbb{R}^l_+$) of the optimal control u_0, study carefully $u_{\varepsilon c}$, the assigned state $y_{\varepsilon c} = y(u_{\varepsilon c})$ and the behaviour of $\mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c})$ dependent on the parameters $\varepsilon > 0$ and $c \in \mathbb{R}^l_+$. Thereby an important role is played by a generalized Green formula giving the solution to a linear boundary value problem for a second order ordinary differential equation whose coefficients are only measurable (cf. [6]). Furthermore we use some separation theorems for sets of convex cones in $\mathbb{R}^n$, which can be found, e.g., in [12] or [3, 2].

There are not very much papers related to the above control problem (**P**); a list of them is given in our talk [8]; article [1] presents a survey of papers published on time optimal control problems for second order ordinary differential equations.

The present paper is written in the spirit of our article [10] and the last paragraphs of the monograph [13]. The contents of the sections is completely described by their headlines.

2. Notations and Assumptions

Nearly all of the notations used throughout the paper are more or less standard. So we denote by $\mathbb{R}^n$, $n \geqslant 1$, the n-dimensional Euclidean space with the norm

$\|\cdot\|$. $\mathbb{R}^n_+$ is the set of all n-dimensional nonnegative vectors. All vectors are to be understood as column vectors. We denote by $\|\cdot\|_C$ the norm in $C[0,1]$ and by $\|\cdot\|_p$ the norm in $L^p(0,1)$, $1 \leq p \leq \infty$. $H^1(0,1)$ denotes the Sobolev space and $H^1_0(0,1)$ its subspace, whose elements vanish at the ends of the interval $(0,1)$. In $H^1_0(0,1)$ the norm is given by $\|y\|_0 = \|y'\|_2$. Let us recall that $H^1_0(0,1)$ is continuously embedded into $C[0,1]$ and that the two elementary inequalities

$$|y(x)| \leq \|y\|_0 \quad \forall x \in [0,1] \quad \text{and} \quad \|y\|_2 \leq \|y\|_0 \tag{5}$$

hold for all $y \in H^1_0(0,1)$. Furthermore, $f \in CAR$ denotes a function $f\colon (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions and K a generic positive constant.

Concerning the given real-valued functions $a = a(x,s,t)$, $b = b(x,r,s,t)$ and $g_\nu = g_\nu(x,r,s,t)$, $\nu = -m_0, \ldots, n_0$, defined for $x \in (0,1)$, $r \in Q$, $s,t \in \mathbb{R}$ and their partial derivatives with respect to s and t we formulate now the assumptions **A1–A3**; two further assumptions **A4** and **A5** will be given afterwards.

A1: It holds

$$\begin{aligned}
&a, a_s, a_t \in CAR\,,\\
&b(\cdot, u(\cdot), \cdot, \cdot), b_s(\cdot, u(\cdot), \cdot, \cdot), b_t(\cdot, u(\cdot), \cdot, \cdot) \in CAR \quad \forall u \in U_{ad}\,,\\
&g_\nu(\cdot, u(\cdot), \cdot, \cdot), g_{\nu s}(\cdot, u(\cdot), \cdot, \cdot), g_{\nu t}(\cdot, u(\cdot), \cdot, \cdot) \in CAR \quad \forall u \in U_{ad}\,.
\end{aligned}$$

A2a: For each $\lambda > 0$ there is a positive constant $\mu_{1\lambda}$ such that

$$|a(x,s,t)| \leq \mu_{1\lambda}(1 + |t|) \quad \text{for} \quad a.e.\, x \in (0,1)\,, \quad \forall s \in \mathbb{R} \quad \text{with} \quad |s| \leq \lambda\,, \quad \forall t \in \mathbb{R}\,.$$

A2b: There is a positive constant α and for each $\lambda > 0$ a positive constant $\mu_{2\lambda}$ such that

$$\alpha \leq a_t(x,s,t) \quad \text{for} \quad a.e.\, x \in (0,1)\,, \quad \forall s,t \in \mathbb{R}\,,$$

and

$$|a_s(x,s,t)|, a_t(x,s,t) \leq \mu_{2\lambda} \quad \text{for} \quad a.e.\, x \in (0,1)\,, \quad \forall s,t \in \mathbb{R} \quad \text{with} \quad |s| + |t| \leq \lambda\,.$$

A2c: There are two positive constants μ and δ such that

$$\begin{aligned}
&|a_s(x,s,t) - a_s(x,\sigma,\tau)| \leq \mu(|s-\sigma| + |t-\tau|)\,,\\
&|a_t(x,s,t) - a_t(x,\sigma,\tau)| \leq \mu(|s-\sigma| + |t-\tau|)\,,
\end{aligned}$$

for $a.e.\, x \in (0,1)$, $\forall s,t,\sigma,\tau \in \mathbb{R}$ with $|s-\sigma|, |t-\tau| \leq \delta$.

A3a: For each $\lambda > 0$ there is a positive constant $\mu_{3\lambda}$ such that

$$|b(x,r,s,t)|,\ |g_\nu(x,r,s,t)| \le \mu_{3\lambda}(1+|t|^2)$$
$$\text{for } a.e.\, x\in(0,1),\quad \forall\{r,s\}\in Q\times\mathbb{R} \text{ with } |r|+|s|\le\lambda,\quad \forall t\in\mathbb{R}.$$

A3b: For each $\lambda>0$ there are two functions $h_{1\lambda}\in L^1(0,1)$ and $h_{2\lambda}\in L^2(0,1)$ such that

$$|b_s(x,r,s,t)|,\ |g_{\nu s}(x,r,s,t)| \le h_{1\lambda}(x),$$
$$|b_t(x,r,s,t)|,\ |g_{\nu t}(x,r,s,t)| \le h_{2\lambda}(x)$$
$$\text{for } a.e.\, x\in(0,1),\quad \forall\{r,s,t\}\in Q\times\mathbb{R}\times\mathbb{R} \text{ with } |r|+|s|+|t|\le\lambda.$$

A3c: For each $\lambda>0$ there are two positive constants $\mu_{4\lambda}$ and δ_λ such that

$$|b_s(x,r,s,t)-b_s(x,r,\sigma,\tau)| \le \mu_{4\lambda}(|s-\sigma|+|t-\tau|),$$
$$|b_t(x,r,s,t)-b_t(x,r,\sigma,\tau)| \le \mu_{4\lambda}(|s-\sigma|+|t-\tau|),$$
$$|g_{\nu s}(x,r,s,t)-g_{\nu s}(x,r,\sigma,\tau)| \le \mu_{4\lambda}(|s-\sigma|+|t-\tau|),$$
$$|g_{\nu t}(x,r,s,t)-g_{\nu t}(x,r,\sigma,\tau)| \le \mu_{4\lambda}(|s-\sigma|+|t-\tau|),$$
$$\text{for } a.e.\, x\in(0,1),\quad \forall r\in Q$$
$$\text{with } |r|\le\lambda, \forall s,t,\sigma,\tau\in\mathbb{R} \text{ with } |s-\sigma|,|t-\tau|\le\delta_\lambda.$$

From these assumptions it follows that

$$a(\cdot,y(\cdot),y'(\cdot))\in L^2(0,1)\quad \forall y\in H^1(0,1),$$
$$b(\cdot,u(\cdot),y(\cdot),y'(\cdot)),\ g_\nu(\cdot,u(\cdot),y(\cdot),y'(\cdot))\in L^1(0,1)\quad \forall u\in U_{ad},$$
$$\forall y\in H^1(0,1).$$

It means that all functionals $\mathcal{T}_\nu$ are well defined for $U_{ad}\times H_0^1(0,1)$ and it makes sense to define the solution to the state equation (2) in the sense of Sobolev. So for any fixed $u\in U_{ad}$ a function $y\in H_0^1(0,1)$ is said to be a (weak) solution to the boundary value problem (2) if and only if

$$\int_0^1 [a(x,y(x),y'(x))z'(x)+b(x,u(x),y(x),y'(x))z(x)]\,dx=0$$
$$\forall z\in H_0^1(0,1). \tag{6}$$

Using a generalized DuBois–Reymond Lemma (cf. [4]) it is easily proved that for fixed $u\in U_{ad}$ a function $y\in H_0^1(0,1)$ is a solution in this sense if and only if there is a constant $k(u)\in\mathbb{R}$ with

$$a(x,y(x),y'(x))=\int_0^x b(\xi,u(\xi),y(\xi),y'(\xi))\,d\xi+k(u)\quad \forall x\in[0,1]. \tag{7}$$

In the whole paper a solution to any linear or nonlinear two point boundary value problem is to be understood in the above sense with the respective integral identity.

Now we are able to give the last two assumptions **A4** and **A5**, which concern the state equation (2).

A4: For each $u \in U_{ad}$ the state equation (2) has a unique solution $y(u) \in H_0^1(0, 1)$ and there is a constant $K > 0$ with

$$\|y(u)\|_0 \leqslant K \quad \forall u \in U_{ad} .$$

A5: For each $\lambda > 0$ there is a positive constant ν_λ such that

$$\int_0^1 \{a_t(x, y, y')z'^2 + [a_s(x, y, y') + b_t(x, u, y, y')]zz' + b_s(x, u, y, y')z^2\} \, dx \geqslant \nu_\lambda \|z\|_0^2$$

$$\forall z \in H^1(0, 1) \quad \text{with} \quad z(0) = 0 \quad \text{or} \quad z(1) = 0 \quad \text{and}$$

$$\forall \{u, y\} \in U_{ad} \times H_0^1(0, 1) \quad \text{for which} \quad y' \in L^\infty(0, 1) ,$$

$$\| |u| \| + \|y'\|_\infty \leqslant \lambda .$$

We finish this section with some further notations. By $\{u_0, y_0\} \in L_m^\infty(0, 1) \times H_0^1(0, 1)$ we denote any optimal pair to the control problem (**P**). For any function an upper index 0 indicates that this function is defined by means of such an optimal pair. For example, we use the abbreviations

$$a^0(x) = a(x, y_0(x), y_0'(x)), \ldots, g_\nu^0(x) = g_\nu(x, u_0(x), y_0(x), y_0'(x))$$

but also

$$a_s^0(x) = a_s(x, y_0(x), y_0'(x)), \ldots, g_{\nu t}^0(x) = g_{\nu t}(x, u_0(x), y_0(x), y_0'(x)) ,$$

where $x \in (0, 1)$.

3. Optimality Conditions

In this section we formulate the Pontryagin minimum principle for the constrained control problem (**P**), from which we derive the linearized (weak) Pontryagin minimum principle within a few steps. The proof of Theorem 1 is the task of the next two sections.

THEOREM 1 (Pontryagin Minimum Principle). *If under the assumptions* **A1–A5** *the pair* $\{u_0, y_0\} \in L_m^\infty(0, 1) \times H_0^1(0, 1)$ *is an optimal solution to problem* (**P**), *then* *there are* m_0+n_0+1 *numbers* $\lambda_\nu \in \mathbb{R}$, *with* $\Sigma_{\nu=-m_0}^{n_0}|\lambda_\nu|>0$ *and* $\lambda_{-m_0}\geq 0,\ldots,\lambda_0\geq 0$, *such that*

$$\left.\begin{array}{l}\sum_{\nu=-m_0}^{n_0}\lambda_\nu[b(x,u,y_0(x),y_0'(x))z_\nu(x)+g_\nu(x,u,y_0(x),y_0'(x))]\geq\\ \sum_{\nu=-m_0}^{n_0}\lambda_\nu[b(x,u_0(x),y_0(x),y_0'(x))z_\nu(x)+g_\nu(x,u_0(x),y_0(x),y_0'(x))]\\ \forall u\in Q\,,\quad for\quad \text{a.e.}\, x\in(0,1)\,,\end{array}\right\} \tag{8}$$

where $z_\nu \in H_0^1(0,1)$, $\nu=-m_0,\ldots,n_0$, *denotes the unique solution to*

$$\left.\begin{array}{l}-\frac{d}{dx}[a_t^0(x)z'(x)+b_t^0(x)z(x)]+[a_s^0(x)z'(x)+b_s^0(x)z(x)]=\\ \frac{d}{dx}g_{\nu t}^0(x)-g_{\nu s}^0(x)\,,\quad x\in(0,1)\,,\quad with\quad z(0)=0\,,\quad z(1)=0\,.\end{array}\right\} \tag{9}$$

As we have said before in the linear differential equation (9) the coefficient $a_t^0, b_t^0, \ldots$ stands for the composed function $a_t(x, y_0(x), u_0'(x))$, $b_t(x, u_0(x), y_0(x), y_0'(x)), \ldots$, respectively. The right hand side of (9) is defined by the integrand g_ν of the functional $\mathcal{T}_\nu$. By definition, $z\in H_0^1(0,1)$ is a solution to (9) provided

$$\left.\begin{array}{l}\int_0^1\{[a_t^0(x)z'(x)+b_t^0(x)z(x)]y'(x)+[a_s^0(x)z'(x)+b_s^0(x)z(x)]y(x)\}\,\mathrm{d}x=\\ \quad -\int_0^1[g_{\nu t}^0(x)y'(x)+g_{\nu s}^0(x)y(x)]\,\mathrm{d}x\quad\forall y\in H_0^1(0,1)\,.\end{array}\right\} \tag{10}$$

Introducing the function

$$\hat{z}_0(x)=\sum_{\nu=-m_0}^{n_0}\lambda_\nu z_\nu(x)\,,\quad x\in(0,1)\,,$$

which is the unique solution to (9) with the right hand side

$$g(\lambda,x)=\sum_{\nu=-m_0}^{n_0}\lambda_\nu\left[\frac{d}{dx}g_{\nu t}^0(x)-g_{\nu s}^0(x)\right],$$

the minimum condition (8) can be written in the possibly more familiar form

$$\begin{array}{l}b(x,u,y_0(x),y_0'(x))\hat{z}_0(x)+\sum_{\nu=-m_0}^{n_0}\lambda_\nu g_\nu(x,u,y_0(x),y_0'(x))\geq\\ b(x,u_0(x),y_0(x),y_0'(x))\hat{z}_0(x)+\sum_{\nu=-m_0}^{n_0}\lambda_\nu g_\nu(x,u_0(x),y_0(x),y_0'(x))\\ \forall u\in Q\,,\quad a.e.\,x\in(0,1)\,.\end{array}$$

To derive the linearized Pontryagin minimum principle we formulate two additional assumptions.

A6: The control set $Q \subset \mathbb{R}^m$ is convex.
A7: For a.e. $x \in (0, 1)$ and for all $s, t \in \mathbb{R}$ the gradients $\nabla_r b(x, \cdot, s, t), \nabla_r g_\nu(x, \cdot, s, t)$ are continuous on Q.

THEOREM 2 (Linearized Pontryagin Minimum Principle). *If under the assumptions* **A1–A7** *the pair* $\{u_0, y_0\} \in L^\infty_m(0, 1) \times H^1_0(0, 1)$ *is an optimal solution to problem* (**P**), *then there are* $m_0 + n_0 + 1$ *numbers* $\lambda_\nu \in \mathbb{R}$, *with* $\Sigma_{\nu=-m_0}^{n_0} |\lambda_\nu| > 0$ *and* $\lambda_{-m_0} \geqslant 0, \ldots, \lambda_0 \geqslant 0$, *such that*

$$\sum_{\nu=-m_0}^{n_0} \lambda_\nu(\nabla_r b(x, u_0(x), y_0(x), y_0'(x))z_\nu(x) + \nabla_r g_\nu(\cdots), u - u_0(x)) \geqslant 0$$
$$\forall u \in Q\,, \quad \text{for} \quad \text{a.e.}\, x \in (0, 1)\,, \tag{11}$$

where $z_\nu \in H^1_0(0, 1)$ *is defined as in Theorem 1.*

The proof of Theorem 2 is very simple. In condition (8) we replace u by the convex linear combination $u_0(x) + \tau(u - u_0(x))$, where $u \in Q$, $\tau \in [0, 1]$ and $x \in (0, 1)$, and put all the terms on one side of the inequality. Afterwards we divide the whole expression by τ and then let τ tend to zero. This procedure yields the desired condition (11).

Theorem 1 generalizes the results of [10], where we have considered an unconstrained control problem for the state equation (2). Under different assumptions and using completely different methods we have proved Theorem 2 for a quasilinear state equation (in which the leading coefficient may also depend on the control parameter) already in [5].

4. Preliminaries

We start the preliminaries with a regularity statement for the solution $y(u) \in H^1_0(0, 1)$ to the state equation (2). Namely, it holds

$$y(u)' \in L^\infty(0, 1) \quad \forall u \in U_{ad}\,, \tag{12}$$

which is an immediate consequence of our assumptions; the proof is given in [10].

We assume now that $\{u_0, y_0\} \in L^\infty_m(0, 1) \times H^1_0(0, 1)$ is optimal to the constrained control problem (**P**). As we said already we want to prove Theorem 1 by means of McShane-variations of the optimal control u_0. To do so, we begin with introducing the two sets

$$\mathscr{X} = \{w \in \mathbb{R}^{l(m+2)}: w = \{u_1, \ldots, u_l, x_1, \ldots, x_l, c_1, \ldots, c_l\}\ ,$$
$$\text{where } l \in \mathbb{N}, u_1, \ldots, u_l \in Q, x_1, \ldots, x_l \in \omega, c = \{c_1, \ldots, c_l\} \in \mathbb{R}^l_+\}\ ,$$

and

$$\mathscr{X}_d = \{w \in \mathscr{X}: x_i \neq x_j \text{ for } i \neq j, i, j = 1, \ldots, l\}\ .$$

Here $\mathbb{N}$ is the set of all natural numbers and ω the set of all Lebesgue points $x \in (0, 1)$ of the functions

$$\left.\begin{aligned} b_{\nu i}(x) &= [b(x, u_i, y_0(x), y_0'(x)) - b^0(x)]z_\nu(x)\ , \\ g_{\nu i}(x) &= g_\nu(x, u_i, y_0(x), y_0'(x)) - g_\nu^0(x)\ , \end{aligned}\right\} \tag{13}$$

$\nu = -m_0, \ldots, n_0, i = 1, \ldots, l; z_\nu \in H_0^1(0, 1)$ denotes the unique solution to (9) (cf. (10) and **A5**). Clearly, $meas\omega = 1$. The restriction to such Lebesgue points will be essential in the proof of the very important Lemma 9.

Let $w \in \mathscr{X}_d$ be fixed. For arbitrary $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, we define

$$E_i^{\varepsilon c} = [x_i, x_i + \varepsilon c_i)\ , \quad i = 1, \ldots, l\ ,$$

and take ε_0 so small that

$$E^{\varepsilon c} = \textstyle\bigcup_{i=1}^l E_i^{\varepsilon c} \subset (0, 1) \quad \text{and} \quad \bigcup_{i=1}^{l-1} (E_i^{\varepsilon c} \cap E_{i+1}^{\varepsilon c}) = \emptyset \quad \forall \varepsilon \in (0, \varepsilon_0)\ . \tag{14}$$

Then, evidently, the function $u_{\varepsilon c}$ defined by

$$u_{\varepsilon c}(x) = \begin{cases} u_i & \text{if } x \in E_i^{\varepsilon c} \\ u_0(x) & \text{if } x \in (0, 1)\backslash E^{\varepsilon c} \end{cases}, \quad \varepsilon \in (0, \varepsilon_0)\ , \tag{15}$$

is an admissible control function called McShane-variation of u_0. By $y_{\varepsilon c} = y(u_{\varepsilon c})$ we denote the assigned solution to state equation (2) (see **A4**). In the following we have to study the behaviour of the pair $\{u_{\varepsilon c}, y_{\varepsilon c}\} \in L_m^\infty(0, 1) \times H_0^1(0, 1)$ and of $\mathscr{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c})$ if one of the defining parameters $\varepsilon \in (0, \varepsilon_0)$ or $c \in \mathbb{R}^l_+$ is changing. Clearly, $u_{\varepsilon c} = u_0$ if $c = 0 \in \mathbb{R}^l_+$.

LEMMA 1. *For any bounded set $\mathscr{C} \in \mathbb{R}^l_+$ there exist two positive constants ε_0 and K such that*

$$|y_{\varepsilon c}(x)| + |y_{\varepsilon c}'(x)| \leq K \quad \textit{for} \quad \textit{a.e.}\ x \in (0, 1)\ , \quad \forall \varepsilon \in (0, \varepsilon_0)\ , \quad \forall c \in \mathscr{C}\ .$$

Proof. First we take $\varepsilon_0 > 0$ so small that (14) is satisfied for any $c \in \mathscr{C}$. Because of (5) and **A4**, we have

$$\|y_{\varepsilon c}\|_C \leq \|y_{\varepsilon c}\|_0 \leq K \quad \forall \varepsilon \in (0, \varepsilon_0)\ , \quad \forall c \in \mathscr{C}\ . \tag{16}$$

Thus, we may take a constant $\lambda > 0$ (not depending on ε and c) such that

$$|u_{\varepsilon c}(x)| + |y_{\varepsilon c}(x)| \leq \lambda \quad \text{for } a.e.\ x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C}. \tag{17}$$

In virtue of the Lagrange formula and equation (7) we find

$$y'_{\varepsilon c}(x) \int_0^1 a_t(x, y_{\varepsilon c}(x), \theta y'_{\varepsilon c}(x))\, d\theta = a(x, y_{\varepsilon c}(x), y'_{\varepsilon c}(x)) - a(x, y_{\varepsilon c}(x), 0)$$
$$= \int_0^x b(\xi, u_{\varepsilon c}(\xi), y_{\varepsilon c}(\xi), y'_{\varepsilon c}(\xi))\, d\xi + k(u_{\varepsilon c}) - a(x, y_{\varepsilon c}(x), 0).$$

Applying assumptions **A2a**, **A2b** and **A3a**, with λ as defined in (17), we obtain

$$|y'_{\varepsilon c}(x)| \leq \alpha^{-1}\Big(\int_0^1 |b(\xi, u_{\varepsilon c}(\xi), y_{\varepsilon c}(\xi), y'_{\varepsilon c}(\xi))|\, d\xi + |k(u_{\varepsilon c})|$$
$$+ |a(x, y_{\varepsilon c}(x), 0)|\Big)$$
$$\leq \alpha^{-1}[\mu_{3\lambda}(1 + \|y_{\varepsilon c}\|_0^2) + |k(u_{\varepsilon c})| + \mu_{1\lambda}] \quad \text{for} \quad a.e.\ x \in (0,1),$$
$$\forall c \in \mathscr{C}. \tag{18}$$

Again using (7), **A2a** and **A3a** we see that

$$|k(u_{\varepsilon c})| \leq \int_0^1 |a(x, y_{\varepsilon c}(x), y'_{\varepsilon c}(x))|\, dx + \int_0^1 |b(x, u_{\varepsilon c}(x), y_{\varepsilon c}(x), y'_{\varepsilon c}(x))|\, dx$$
$$\leq \mu_{1\lambda}(1 + \|y_{\varepsilon c}\|_0) + \mu_{3\lambda}(1 + \|y_{\varepsilon c}\|_0^2) \quad \forall c \in \mathscr{C}.$$

This estimation, together with (16) and (18), yields the desired estimation. ■

Since the next lemma can be proved by straightforward calculations, its proof is omitted.

LEMMA 2. *For each bounded set $\mathscr{C} \in \mathbb{R}^l_+$ there are two positive constants ε_0 and K such that*

$$\|u_{\varepsilon c} - u_{\varepsilon d}\|_2^2 \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i| \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c, d \in \mathscr{C}.$$

Mainly using assumption **A5** it is not hard to get a similar estimation for $\|y_{\varepsilon c} - y_{\varepsilon d}\|_0$ (see below). But since for $|y'_{\varepsilon c}(\cdot) - y'_{\varepsilon d}(\cdot)|$ a *pointwise* estimation is required we need much more efforts.

Let $\mathscr{C} \subset \mathbb{R}^l_+$ be bounded and $\varepsilon_0 > 0$ so small that (14) holds for each $c \in \mathscr{C}$. For arbitrary $\varepsilon \in (0, \varepsilon_0)$ and $c, d \in \mathscr{C}$ we define some auxiliary functions by setting

$$y(\theta)(x) = y_{\varepsilon d}(x) + \theta(y_{\varepsilon c}(x) - y_{\varepsilon d}(x)), \quad \theta \in [0,1], \tag{19}$$

$$\left.\begin{aligned}
a_1(\varepsilon, c, d)(x) &= \int_0^1 a_t(x, y(\theta)(x), y(\theta)'(x))\, d\theta, \\
a_2(\varepsilon, c, d)(x) &= \int_0^1 a_s(x, y(\theta)(x), y(\theta)'(x))\, d\theta, \\
b_1(\varepsilon, c, d)(x) &= \int_0^1 b_t(x, u_{\varepsilon c}(x), y(\theta)(x), y(\theta)'(x))\, d\theta, \\
b_2(\varepsilon, c, d)(x) &= \int_0^1 b_s(x, u_{\varepsilon c}(x), y(\theta)(x), y(\theta)'(x))\, d\theta, \\
g_{\nu 1}(\varepsilon, c, d)(x) &= \int_0^1 g_{\nu t}(x, u_{\varepsilon c}(x), y(\theta)(x), y(\theta)'(x))\, d\theta, \\
g_{\nu 2}(\varepsilon, c, d)(x) &= \int_0^1 g_{\nu s}(x, u_{\varepsilon c}(x), y(\theta)(x), y(\theta)'(x))\, d\theta,
\end{aligned}\right\} \tag{20}$$

and

$$\left.\begin{aligned}
\Delta b(\varepsilon, c, d)(x) &= b(x, u_{\varepsilon c}(x), y_{\varepsilon d}(x), y'_{\varepsilon d}(x)) - b(x, u_{\varepsilon d}(x), y_{\varepsilon d}(x), y'_{\varepsilon d}(x)), \\
\Delta g_\nu(\varepsilon, c, d)(x) &= g_\nu(x, u_{\varepsilon c}(x), y_{\varepsilon d}(x), y'_{\varepsilon d}(x)) - g_\nu(x, u_{\varepsilon d}(x), y_{\varepsilon d}(x), y'_{\varepsilon d}(x)),
\end{aligned}\right\} \tag{21}$$

where $x \in (0,1)$ and $\nu = -m_0, \ldots, n_0$. For arbitrary $\varepsilon \in (0, \varepsilon_0)$ and $c, d \in \mathscr{C}$ these functions have the following properties:

$$\left.\begin{aligned}
&a_1(\varepsilon, c, d), a_2(\varepsilon, c, d) \in L^\infty(0,1) && \text{with} \\
&\alpha \leq a_1(\varepsilon, c, d)(x) \leq K, |a_2(\varepsilon, c, d)(x)| \leq K, && \text{for} \quad a.e.\, x \in (0,1),
\end{aligned}\right\} \tag{22}$$

$$b_1(\varepsilon, c, d), g_{\nu 1}(\varepsilon, c, d) \\ \in L^2(0,1) \quad \text{with} \quad \|b_1(\varepsilon, c, d)\|_2, \|g_{\nu 1}(\varepsilon, c, d)\|_2 \leq K, \tag{23}$$

$$b_2(\varepsilon, c, d), g_{\nu 2}(\varepsilon, c, d) \\ \in L^1(0,1) \quad \text{with} \quad \|b_2(\varepsilon, c, d)\|_1, \|g_{\nu 2}(\varepsilon, c, d)\|_1 \leq K, \tag{24}$$

$$\Delta b(\varepsilon, c, d), \Delta g_\nu(\varepsilon, c, d) \\ \in L^1(0,1) \quad \text{with} \quad \|\Delta b(\varepsilon, c, d)\|_1, \|\Delta g_\nu(\varepsilon, c, d)\|_1 \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i|. \tag{25}$$

Here K denotes a positive constant neither depending on $\varepsilon \in (0, \varepsilon_0)$ nor on $c, d \in \mathscr{C}$. (22)–(25) are more or less simple consequences of **A1**, (14), Lemma 1 and the growth conditions in **A2b** and **A3b**. As an example let us prove in detail only the estimation for $\|\Delta b(\varepsilon, c, d)\|_1$ in (25). By definition of $u_{\varepsilon c}$ and $u_{\varepsilon d}$, it holds

$$u_{\varepsilon c}(x) = u_{\varepsilon d}(x) = u_0(x) \quad \text{if} \quad x \in (0,1) \backslash (E^{\varepsilon c} \cup E^{\varepsilon d})$$

and therefore

$$\|\Delta b(\varepsilon, c, d)\|_1 = \int_{E^{\varepsilon c} \cup E^{\varepsilon d}} |\Delta b(\varepsilon, c, d)| \, dx = \sum_{i=1}^{l} \int_{E_i^{\varepsilon c} \cup E_i^{\varepsilon d}} |\Delta b(\varepsilon, c, d)| \, dx \, .$$

Because of

$$u_{\varepsilon c}(x) = u_{\varepsilon d}(x) = u_i(x) \quad \text{if} \quad x \in E_i^{\varepsilon c} \cap E_i^{\varepsilon d} \, , \quad i = 1, \ldots, l \, ,$$

it remains

$$\|\Delta b(\varepsilon, c, d)\|_1 = \sum_{i=1}^{l} \int_{E_i} |\Delta b(\varepsilon, c, d)| \, dx$$

with

$$E_i = (E_i^{\varepsilon c} \cup E_i^{\varepsilon d}) \backslash (E^{\varepsilon c} \cap E^{\varepsilon d}) = \begin{cases} [x_i + \varepsilon c_i, x_i + \varepsilon d_i] & \text{if} \quad c_i < d_i \\ \emptyset & \text{if} \quad c_i = d_i \, , \\ [x_i + \varepsilon d_i, x_i + \varepsilon c_i] & \text{if} \quad c_i > d_i \end{cases} \quad i = 1, \ldots, l \, .$$

In virtue of Lemma 1 we can choose a $\lambda > 0$, such that

$$|u_0(x)| + |y_{\varepsilon d}(x)| + |y'_{\varepsilon d}(x)| \leqslant \lambda \quad \text{and} \quad |u_i| + |y_{\varepsilon d}(x)| + |y'_{\varepsilon d}(x)| \leqslant \lambda$$

hold for a.e. $x \in (0,1)$, all $\varepsilon \in (0, \varepsilon_0)$, $d \in \mathscr{C}$ and any $i = 1, \ldots, l$. Then, by assumption **A3a**, it follows

$$\int_{E_i} |\Delta b(\varepsilon, c, d)| \, dx \leqslant 2\mu_{3\lambda} \int_{E_i} (1 + |y'_{\varepsilon c}(x)|^2) \, dx$$
$$\leqslant 2\mu_{3\lambda} \varepsilon \, (1 + \lambda^2) |c_i - d_i| \, , \quad i = 1, \ldots, l \, ,$$

that means, the desired estimation in (25) holds with $K = 2\mu_{3\lambda}(1 + \lambda^2)$.

Using the auxiliary functions defined in (20), (21) we formulate the parametric linear boundary value problem

$$\left.\begin{aligned} & -\frac{d}{dx}[a_1(\varepsilon, c, d)(x)\rho'(x) + a_2(\varepsilon, c, d)(x)\rho(x)] \\ & + [b_1(\varepsilon, c, d)(x)\rho'(x) + b_2(\varepsilon, c, d)(x)\rho(x)] = \\ & - \Delta b(\varepsilon, c, d)(x) \, , \quad x \in (0,1) \, , \quad \text{with} \quad \rho(0) = 0 \, , \quad \rho(1) = 0 \, , \end{aligned}\right\} \tag{26}$$

for which $\rho \in H_0^1(0,1)$ is said to be a solution if

$$\left.\begin{aligned} &\int_0^1 \{[a_1(\varepsilon,c,d)\rho' + a_2(\varepsilon,c,d)\rho]z' + [b_1(\varepsilon,c,d)\rho' \\ &\quad + b_2(\varepsilon,c,d)\rho]z\}\,dx = -\int_0^1 \Delta b(\varepsilon,c,d)z\,dx \qquad \forall z \in H_0^1(0,1)\,. \end{aligned}\right\} \tag{27}$$

Again taking into account Lemma 1 we find a $\lambda > 0$, such that for $z = z(\theta)$ defined in (19) it holds

$$\| |u_{\varepsilon c}| \|_\infty + \|y(\theta)'\|_\infty \le \lambda \quad \forall \theta \in [0,1] \quad \forall c, d \in \mathscr{C}\,.$$

Therefore, in assumption **A5** we can replace u by $u_{\varepsilon c}$ and y by $y(\theta)$, $\theta \in [0,1]$, respectively. If we do so and if we integrate the resulting inequality with respect to $\theta \in [0,1]$, we see that the boundary value problem (26) is coercive on $H_0^1(0,1)$, where the coerciveness constant does not depend on the parameters $\varepsilon \in (0, \varepsilon_0)$ and $c, d \in \mathscr{C}$ (but of course on $\mathscr{C} \subset \mathbb{R}_+^l$). This means, due to the generalized Lax–Milgram-Theorem for all these parameters the boundary value problem (26) has a unique solution $\rho(\varepsilon,c,d) \in H_0^1(0,1)$. We claim

$$\rho(\varepsilon,c,d) = y_{\varepsilon c} - y_{\varepsilon d} \quad \forall \varepsilon \in (0,\varepsilon_0) \quad \forall c,d \in \mathscr{C}\,. \tag{28}$$

Indeed, by the respective definitions of $y_{\varepsilon c}$ and $y_{\varepsilon d}$, for any $z \in H_0^1(0,1)$ we have

$$\begin{aligned} 0 = &\int_0^1 [a(x, y_{\varepsilon c}, y'_{\varepsilon c}) - a(x, y_{\varepsilon d}, y'_{\varepsilon d})]z'\,dx \\ &+ \int_0^1 [b(x, u_{\varepsilon c}, y_{\varepsilon c}, y'_{\varepsilon c}) - b(x, u_{\varepsilon c}, y_{\varepsilon d}, y'_{\varepsilon d})]z\,dx \\ &+ \int_0^1 [b(x, u_{\varepsilon c}, y_{\varepsilon d}, y'_{\varepsilon d}) - b(x, u_{\varepsilon d}, y_{\varepsilon d}, y'_{\varepsilon d})]z\,dx\,. \end{aligned}$$

Applying Lagrange formula to the first two terms and using the functions $a_1(\varepsilon,c,d), \ldots, b_2(\varepsilon,c,d)$ and $\Delta(\varepsilon,c,d)$ defined in (20) and (21), respectively, we derive

$$\begin{aligned} 0 = &\int_0^1 [a_1(\varepsilon,c,d)(y'_{\varepsilon c} - y'_{\varepsilon d}) + a_2(\varepsilon,c,d)(y_{\varepsilon c} - y_{\varepsilon d})]z'\,dx \\ &+ \int_0^1 [b_1(\varepsilon,c,d)(y'_{\varepsilon c} - y'_{\varepsilon d}) + b_2(\varepsilon,c,d)(y_{\varepsilon c} - y_{\varepsilon d})]z\,dx \\ &+ \int_0^1 \Delta b(\varepsilon,c,d)z\,dx \quad \forall z \in H_0^1(0,1)\,, \end{aligned}$$

which already proves (28).

Due to the uniform coercivity of the boundary value problem (26) and because of (25) we easily see that there is a positive constant K, such that

$$\|y_{\varepsilon c} - y_{\varepsilon d}\|_0 \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i| \quad \forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,,$$

from which, noting (5), we obtain

$$|y_{\varepsilon c}(x) - y_{\varepsilon d}(x)| \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i| \quad \text{for a.e. } x \in (0, 1)\,,$$
$$\forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,. \tag{29}$$

Unfortunately, we cannot argue in the same way to get an analogous estimation for $|y'_{\varepsilon c}(x) - y'_{\varepsilon d}(x)|$. However, we may use the results in [6], which allows to represent the solution to (26) and its derivative by means of a generalized parametric Green function $G(\varepsilon, c, d) = G(\varepsilon, c, d)(x, \xi)$, having the properties

$$G(\varepsilon, c, d), G(\varepsilon, c, d)_x \in L^\infty((0, 1) \times (0, 1)) \quad \text{with}$$
$$|G(\varepsilon, c, d)(x, \xi)|, |G(\varepsilon, c, d)_x(x, \xi)| \leq K \quad \text{for a.e. } x, \xi \in (0, 1)\,,$$
$$\forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,,$$

in the form

$$\rho(\varepsilon, c.d)(x) = -\int_0^1 G(\varepsilon, c, d)(x, \xi)\, \Delta b(\varepsilon, c, d)(\xi)\, \mathrm{d}\xi\,, \quad x \in (0, 1)\,,$$

$$\rho(\varepsilon, c.d)'(x) = -\int_0^1 G(\varepsilon, c, d)_x(x, \xi)\, \Delta b(\varepsilon, c, d)(\xi)\, \mathrm{d}\xi\,, \quad x \in (0, 1)\,.$$

Because of (25) and (28) from the first formula we obtain again estimation (29) and from the second one we derive the needed estimation for $|y'_{\varepsilon c}(x) - y'_{\varepsilon d}(x)|$, namely

$$|y'_{\varepsilon c}(x) - y'_{\varepsilon d}(x)| \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i| \quad \text{for a.e. } x, \xi \in (0, 1)\,,$$
$$\forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,.$$

We summarize our findings in the next lemma.

LEMMA 3. *Let* $\mathscr{C} \subset \mathbb{R}^l_+$ *be bounded,* $\varepsilon_0 > 0$ *so small that* (14) *holds for all* $c \in \mathscr{C}$ *and let* $y_{\varepsilon c} = y(u_{\varepsilon c}) \in H^1_0(0, 1)$ *and* $y_{\varepsilon d} = y(u_{\varepsilon d}) \in H^1_0(0, 1)$ *be the respective solutions to* (2) *for any* $c, d \in \mathscr{C}$. *Then* $y_{\varepsilon c} - y_{\varepsilon d}$ *is the unique solution to* (26) *and there exists a positive constant K such that*

$$|y_{\varepsilon c}(x) - y_{\varepsilon d}(x)| + |y'_{\varepsilon c}(x) - y'_{\varepsilon d}(x)| \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i| \quad \textit{for} \quad \text{a.e. } x \in (0,1)\,,$$
$$\forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c, d \in \mathscr{C}\,.$$

This Lemma 3 can be applied to $\mathscr{C} \cup \{0\} \subset \mathbb{R}^l_+$ provided $\mathscr{C}$ is bounded. So let us take $c \in \mathscr{C}$ and $d = 0$. Then we have $u_{\varepsilon d} = u_0$ and $y_{\varepsilon d} = y_0$. Using the auxiliary functions defined in (19)–(21) we introduce the following abbreviations:

$$\left.\begin{aligned} &a_1^{\varepsilon c} = a_1(\varepsilon, c, 0), a_2^{\varepsilon c} = a_2(\varepsilon, c, 0), \ldots, g_{\nu 2}^{\varepsilon c} = g_{\nu 2}(\varepsilon, c, 0)\,, \\ &\Delta b^{\varepsilon c} = \Delta b(\varepsilon, c, 0) \quad \text{and} \quad \Delta g_\nu^{\varepsilon c} = \Delta g_\nu(\varepsilon, c, 0)\,. \end{aligned}\right\} \tag{30}$$

The boundary value problem (26) reduces eventually to

$$\left.\begin{aligned} &-\frac{d}{dx}[a_1^{\varepsilon c}(x)\rho'(x) + a_2^{\varepsilon c}(x)\rho(x)] + [b_1^{\varepsilon c}(x)\rho'(x) + b_2^{\varepsilon c}(x)\rho(x)] = -\Delta b^{\varepsilon c}(x)\,, \\ &x \in (0,1)\,, \quad \text{with} \quad \rho(0) = 0\,, \quad \rho(1) = 0\,, \end{aligned}\right\} \tag{31}$$

and the variational equality (27) to

$$\int_0^1 \{[a_1^{\varepsilon c}\rho' + a_2^{\varepsilon c}\rho]z' + [b_1^{\varepsilon c}\rho' + b_2^{\varepsilon c}\rho]z\}\,dx = -\int_0^1 \Delta b^{\varepsilon c} z\,dx$$
$$\forall z \in H_0^1(0,1)\,. \tag{32}$$

That means, in the special case considered here Lemma 3 reads as follows.

LEMMA 4. *For any bounded set* $\mathscr{C} \subset \mathbb{R}^l_+$ *there exists a positive constant* ε_0 *such that*

$$\rho_{\varepsilon c} = y_{\varepsilon c} - y_0 \in H_0^1(0,1) \quad \forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,,$$

is the unique solution to (31) *and there is another positive constant K such that*

$$|\rho_{\varepsilon c}(x)| + |\rho'_{\varepsilon c}(x)| \leq \varepsilon K \quad \text{for a.e. } x \in (0,1)\,, \quad \forall \varepsilon \in (0, \varepsilon_0)\,, \quad \forall c \in \mathscr{C}\,.$$

For later purposes we have to calculate the limits of the functions $a_1^{\varepsilon c}, \ldots, g_{\nu 2}^{\varepsilon c}$ as ε tends to zero. These limits are given in the next lemma.

LEMMA 5. *If* $\varepsilon \to +0$, *then*

$$a_1^{\varepsilon c} \to a_t^0\,, \quad a_2^{\varepsilon c} \to a_s^0 \quad \textit{in } L^2(0,1)\,,$$
$$b_1^{\varepsilon c} \to b_t^0\,, \quad b_2^{\varepsilon c} \to b_s^0\,, \quad g_{\nu 1}^{\varepsilon c} \to g_{\nu t}^0\,, \quad g_{\nu 2}^{\varepsilon c} \to g_{\nu s}^0 \quad \textit{in } L^1(0,1)$$

uniformly on any bounded set $\mathscr{C} \subset \mathbb{R}^l_+$.

Proof. Let $\varepsilon_0 > 0$ be so small that (14) holds for any $\varepsilon \in (0, \varepsilon_0)$ and that, hence, $\rho_{\varepsilon c} = y_{\varepsilon c} - y_0$ and $a_1^{\varepsilon c}, \ldots, g_{\nu 2}^{\varepsilon c}$ are defined for all $\varepsilon \in (0, \varepsilon_0)$ and all $c \in \mathscr{C}$. We prove only the first and the last statement; the other statements are proved analogously.

By definition of $a_1^{\varepsilon c}$, we have

$$\left.\begin{aligned} &|a_1^{\varepsilon c}(x) - a_t^0(x)| \leq \int_0^1 |a_t(x, y_0(x) + \theta\rho_{\varepsilon c}(x), y_0'(x) + \theta\rho_{\varepsilon c}'(x)) - a_t^0(x)|\, d\theta \\ &\quad \text{for a.e. } x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C}. \end{aligned}\right\} \tag{33}$$

In virtue of Lemma 4 we can take such a small $\varepsilon \in (0, \varepsilon_0)$ that

$$|\rho_{\varepsilon c}(x)|, |\rho_{\varepsilon c}'(x)| \leq \delta \quad \text{for a.e. } x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C},$$

where $\delta > 0$ is taken out from assumption **A2c**. Applying this assumption to (33) we get

$$\begin{aligned} &|a_1^{\varepsilon c}(x) - a_t^0(x)| \leq \mu(|\rho_{\varepsilon c}(x)| + |\rho_{\varepsilon c}'(x)|) \quad \text{for a.e. } x \in (0,1), \\ &\quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C}, \end{aligned}$$

from which, again because of Lemma 4, the first assertion follows.

For proving the last assertion we take $\lambda > 0$ so large that

$$|u_{\varepsilon c}(x)| \leq \lambda \quad \text{for a.e. } x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C},$$

and

$$\begin{aligned} &|u_0(x)| + |y_0(x)| + |y_0'(x)|, |u_i| + |y_0(x)| + |y_0'(x)| \leq \lambda \\ &\quad \text{for a.e. } x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall c \in \mathscr{C}, \quad i = 1, \ldots, l. \end{aligned}$$

(Recall, by (12), we have $y_0' \in L^\infty(0,1)$.) Then, taking $\varepsilon \in (0, \varepsilon_0)$ so small that

$$|\rho_{\varepsilon c}(x)|, |\rho_{\varepsilon c}'(x)| \leq \delta_\lambda \quad \text{for a.e. } x \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0) \quad \forall c \in \mathscr{C},$$

where now $\delta_\lambda > 0$ is the same constant as in assumption **A3c**, we obtain the estimate

$$\begin{aligned} \|g_{\nu 2}^{\varepsilon c} - g_{\nu s}^0\|_1 &\leq \int_0^1 |g_{\nu 2}^{\varepsilon c}(x) - g_{\nu s}(x, u_{\varepsilon c}(x), y_0(x), y_0'(x))|\, dx \\ &\quad + \int_0^1 |g_{\nu s}(x, u_{\varepsilon c}(x), y_0(x), y_0'(x)) - g_{\nu s}^0(x)|\, dx \\ &\leq \int_0^1 \int_0^1 |g_{\nu s}(x, u_{\varepsilon c}(x), y_0(x) + \theta\rho_{\varepsilon c}(x), y_0'(x) + \theta\rho_{\varepsilon c}'(x)) \end{aligned}$$

$$- g_{\nu s}(x, u_{\varepsilon c}(x), y_0(x), y_0'(x))| \,d\theta \,dx$$
$$+ \sum_{i=1}^{l} \int_{x_i}^{x_i+\varepsilon c_i} |g_{\nu s}(x, u_i, y_0(x), y_0'(x)) - g_{\nu s}^0(x)| \,dx$$
$$\leq \mu_{4\lambda} \int_0^1 (|\rho_{\varepsilon c}(x)| + |\rho_{\varepsilon c}'(x)|) \,dx + 2 \sum_{i=1}^{l} \int_{x_i}^{x_i+\varepsilon c_i} h_{1\lambda}(x) \,dx\,,$$

where in the last step assumptions **A3c** and **A3b** were used. This estimate implies the last assertion of the lemma. ■

Now we are going to investigate the functionals $\mathcal{T}_\nu$. First we prove the continuity of $\mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c})$ with respect to $c \in \mathbb{R}_+^l$.

LEMMA 6. *For any bounded set $\mathscr{C} \subset \mathbb{R}_+^l$ there is a positive constant $\varepsilon_0 > 0$ such that for arbitrarily fixed $\varepsilon \in (0, \varepsilon_0)$ the functions $\phi_\nu(c) = \mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c})$, $\nu = -m_0, \dots, n_0$, are continuous on $\mathscr{C}$.*

Proof. Let $\varepsilon_0 > 0$ be taken so small that (14) holds for all $c \in \mathscr{C}$. Then, for fixed $\varepsilon \in (0, \varepsilon_0)$, $c, d \in \mathscr{C}$ and $\nu = -m_0, \dots, n_0$ we have the identity

$$\begin{aligned}
&\mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c}) - \mathcal{T}_\nu(u_{\varepsilon d}, y_{\varepsilon d}) \\
&\quad = \int_0^1 [g_\nu(\cdot, u_{\varepsilon c}, y_{\varepsilon c}, y_{\varepsilon c}') - g_\nu(\cdot, u_{\varepsilon c}, y_{\varepsilon d}, y_{\varepsilon d}')] \,dx \\
&\qquad + \int_0^1 [g_\nu(\cdot, u_{\varepsilon c}, y_{\varepsilon d}, y_{\varepsilon d}') - g_\nu(\cdot, u_{\varepsilon d}, y_{\varepsilon d}, y_{\varepsilon d}')] \,dx \\
&\quad = \int_0^1 [g_{\nu 1}(\varepsilon, c, d)(y_{\varepsilon c}' - y_{\varepsilon d}') + g_{\nu 2}(\varepsilon, c, d)(y_{\varepsilon c} - y_{\varepsilon d}) + \Delta g_\nu(\varepsilon, c, d)] \,dx
\end{aligned} \tag{34}$$

from which, in virtue of Lemma 3 and (23)–(25), we obtain the estimation

$$|\mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c}) - \mathcal{T}_\nu(u_{\varepsilon d}, y_{\varepsilon d})| \leq \varepsilon K \sum_{i=1}^{l} |c_i - d_i|$$

showing the wanted continuity of ϕ_ν on $\mathscr{C}$. ■

We introduce some further abbreviations. For any $w \in \{u_1, \dots, u_l, x_1, \dots, x_l, c_1, \dots, c_l\} \in \mathscr{X}$ we put

$$\begin{aligned}
\mathcal{T}_\nu(w) = \sum_{i=1}^{l} c_i\{&[b(x_i, u_i, y_0(x_i), y_0'(x_i)) - b^0(x_i)]z_\nu(x_i) \\
&+ [g_\nu(x_i, u_i, y_0(x_i), y_0'(x_i)) - g_\nu^0(x_i)]\}\,, \quad \nu = -m_0, \dots, n_0\,,
\end{aligned} \tag{35}$$

and for any $w \in \{u_1, \ldots, u_l, x_1, \ldots, x_l, c_1, \ldots, c_l\} \in \mathscr{X}_d$ and $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small

$$\mathscr{T}_\nu(\varepsilon, w) = \mathscr{T}_{\nu 1}(\varepsilon, w) + \mathscr{T}_{\nu 2}(\varepsilon, w)\,, \quad \nu = -m_0, \ldots, n_0\,,$$

$$\begin{aligned}\mathscr{T}_{\nu 1}(\varepsilon, w) = \sum_{i=1}^{l} \Big\{ & \int_{x_i}^{x_i+\varepsilon c_i} [b(x, u_i, y_0(x), y_0'(x)) - b^0(x)] z_\nu(x)\,\mathrm{d}x \\ & - \varepsilon c_i [b(x_i, u_i, y_0(x_i), y_0'(x_i)) - b^0(x_i)] z_\nu(x_i) \Big\} \\ + \sum_{i=1}^{l} \Big\{ & \int_{x_i}^{x_i+\varepsilon c_i} [g_\nu(x, u_i, y_0(x), y_0'(x)) - g_\nu^0(x)] z_\nu(x)\,\mathrm{d}x \\ & - \varepsilon c_i [g_\nu(x_i, u_i, y_0(x_i), y_0'(x_i)) - g_\nu^0(x_i)] z_\nu(x_i) \Big\}\,,\end{aligned}$$

$$\begin{aligned}\mathscr{T}_{\nu 2}(\varepsilon, w) = & \int_0^1 (a_1^{\varepsilon c} - a_t^0) z_\nu' \rho_{\varepsilon c}'\,\mathrm{d}x + \int_0^1 (a_2^{\varepsilon c} - a_s^0) z_\nu' \rho_{\varepsilon c}\,\mathrm{d}x \\ & + \int_0^1 (b_1^{\varepsilon c} - b_t^0) z_\nu \rho_{\varepsilon c}'\,\mathrm{d}x + \int_0^1 (b_2^{\varepsilon c} - b_s^0) z_\nu \rho_{\varepsilon c}\,\mathrm{d}x \\ & + \int_0^1 (g_{\nu 1}^{\varepsilon c} - g_{\nu t}^0) \rho_{\varepsilon c}'\,\mathrm{d}x + \int_0^1 (g_{\nu 2}^{\varepsilon c} - g_{\nu s}^0) \rho_{\varepsilon c}\,\mathrm{d}x\,.\end{aligned}$$

Here, since we have different types of arguments, one cannot mix $\mathscr{T}_\nu(\varepsilon, w)$ up with $\mathscr{T}_\nu(u, y)$. In both (35) and (36) $z_\nu \in H_0^1(0, 1)$ denotes the unique solution to the boundary value problem (9). Note $\mathscr{T}_\nu(w)$ is defined for arbitrary $w \in \mathscr{X}$, but $\mathscr{T}_\nu(\varepsilon, w)$ only for $w \in \mathscr{X}_d$ (as well as $u_{\varepsilon c}$). Their importance will be clear after the next two lemmas.

LEMMA 7. *For any $w \in \mathscr{X}_d$ it holds*

$$\mathscr{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c}) = \mathscr{T}_\nu(u_0, y_0) + \varepsilon \mathscr{T}_\nu(w) + \mathscr{T}_\nu(\varepsilon, w)\,,$$
$$\nu = -m_0, \ldots, n_0\,, \quad \forall \varepsilon \in (0, \varepsilon_0)\,,$$

with $\varepsilon_0 > 0$ sufficiently small.

Proof. Let $w \in \mathscr{X}_d$ be given and $\varepsilon_0 > 0$ so small that (14) is fulfilled. Then, because of (34) (with $d = 0$), for any $\nu = -m_0, \ldots, n_0$ and $\varepsilon \in (0, \varepsilon_0)$ we have

$$\mathscr{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c}) - \mathscr{T}_\nu(u_0, y_0) = \int_0^1 [g_{\nu 1}^{\varepsilon c} \rho_{\varepsilon c}' + g_{\nu 2}^{\varepsilon c} \rho_{\varepsilon c}]\,\mathrm{d}x + \int_0^1 \Delta g_\nu^{\varepsilon c}\,\mathrm{d}x\,,$$

where $g_{\nu 1}^{\varepsilon c}$, $g_{\nu 2}^{\varepsilon c}$ and $\Delta g_\nu^{\varepsilon c}$ are defined in (30). Since $z_\nu \in H_0^1(0, 1)$ is a solution to the boundary value problem (9), it holds

$$0 = -\int_0^1 \{[a_t^0 z_\nu' + b_t^0 z_\nu] \rho_{\varepsilon c}' + [a_s^0 z_\nu' + b_s^0 z_\nu] \rho_{\varepsilon c}\}\,\mathrm{d}x - \int_0^1 [g_{\nu t}^0 \rho_{\varepsilon c}' + g_{\nu s}^0 \rho_{\varepsilon c}]\,\mathrm{d}x$$

(cf. (10)). By Lemma 4, function $\rho_{\varepsilon c} \in H_0^1(0,1)$ is a solution to the boundary value problem (31), which implies

$$0 = \int_0^1 \{[a_1^{\varepsilon c}\rho'_{\varepsilon c} + a_2^{\varepsilon c}\rho_{\varepsilon c}]z'_\nu + [b_1^{\varepsilon c}\rho'_{\varepsilon c} + b_2^{\varepsilon c}\rho_{\varepsilon c}]z_\nu\}\, dx + \int_0^1 \Delta b^{\varepsilon c} z_\nu \, dx$$

(cf. (32)). Thus, summing up the last three relations we find

$$\mathcal{T}_\nu(u_{\varepsilon c}, y_{\varepsilon c}) - \mathcal{T}_\nu(u_0, y_0) = \int_0^1 [\Delta b^{\varepsilon c} z_\nu + \Delta g_\nu^{\varepsilon c}]\, dx + \mathcal{T}_{\nu 2}(\varepsilon, w)\,.$$

Furthermore, by definition of $\Delta g_\nu^{\varepsilon c}$ and $\Delta b^{\varepsilon c}$ (cf. (30), (21)), we have

$$\begin{aligned}\int_0^1 [\Delta b^{\varepsilon c} z_\nu + \Delta g_\nu^{\varepsilon c}]\, dx &= \sum_{i=1}^{l} \int_{x_i}^{x_i+\varepsilon c_i} [b(x, u_i, y_0(x), y'_0(x)) - b^0(x)] z_\nu(x)\, dx \\ &\quad + \sum_{i=1}^{l} \int_{x_i}^{x_i+\varepsilon c_i} [g_\nu(x, u_i, y_0(x), y'_0(x)) - g_\nu^0(x)]\, dx \\ &= \varepsilon \mathcal{T}_\nu(w) + \mathcal{T}_{\nu 1}(\varepsilon, w)\,.\end{aligned}$$

Thereby, the lemma is proved. ■

LEMMA 8. *Let $\mathscr{C} \subset \mathbb{R}_+^l$ be bounded and $\varepsilon_0 > 0$ so small that* (14) *is valid for any $c \in \mathscr{C}$. Then for any $\eta > 0$ there is a $\delta_0 = \delta_0(\mathscr{C}, \eta) \in (0, \varepsilon_0)$ such that*

$$\varepsilon^{-1}|\mathcal{T}_\nu(\varepsilon, w)| \leq \eta \quad \forall \varepsilon \in (0, \delta_0)\,, \quad \forall c \in \mathscr{C}\,, \quad \nu = -m_0, \ldots, n_0\,.$$

Proof. Because of (36) the lemma is proved after showing the assertion is true for both $\mathcal{T}_{\nu 1}$ and $\mathcal{T}_{\nu 2}$. Using the notations introduced in (13) for $\mathcal{T}_{\nu 1}(\varepsilon, w)$, $\nu = -m_0, \ldots, n_0$, we can write

$$\begin{aligned}\mathcal{T}_{\nu 1}(\varepsilon, w) &= \sum_{i=1}^{l} \left[\int_{x_i}^{x_i+c_i} b_{\nu i}(x)\, dx - \varepsilon\, c_i b_{\nu i}(x_i)\right] \\ &\quad + \sum_{i=1}^{l} \left[\int_{x_i}^{x_i+\varepsilon c_i} g_{\nu i}(x)\, dx - \varepsilon c_i g_{\nu i}(x_i)\right],\end{aligned}$$

from which it follows

$$\begin{aligned}\varepsilon^{-1}|\mathcal{T}_{\nu 1}(\varepsilon, w)| &\leq \sum_{i=1}^{l} \varepsilon^{-1} \int_{x_i}^{x_i+\varepsilon c_i} |b_{\nu i}(x) - b_{\nu i}(x_i)|\, dx \\ &\quad + \sum_{i=1}^{l} \varepsilon^{-1} \int_{x_i}^{x_i+\varepsilon c_i} |g_{\nu i}(x) - g_{\nu i}(x_i)|\, dx\end{aligned}$$

valid for any $\varepsilon \in (0, \varepsilon_0)$ and $c \in \mathscr{C}$. In virtue of boundedness of $\mathscr{C} \subset \mathbb{R}_+^l$ we find a constant $k>0$ with $k \geqslant c_i$ for each $i=1,\dots,l$ and each $c=\{c_1,\dots,c_l\} \in \mathscr{C}$. That means, we get

$$\begin{aligned}\varepsilon^{-1}|\mathscr{T}_{\nu 1}(\varepsilon, w)| \leqslant k \sum_{i=1}^{l} \frac{1}{\varepsilon k} \int_{x_i}^{x_i+\varepsilon k} |b_{\nu i}(x) - b_{\nu i}(x_i)|\,\mathrm{d}x \\ + k \sum_{i=1}^{l} \frac{1}{\varepsilon k} \int_{x_i}^{x_i+\varepsilon k} |g_{\nu i}(x) - g_{\nu i}(x_i)|\,\mathrm{d}x\end{aligned}$$

valid for any $\varepsilon \in (0, \varepsilon_0)$ and $c \in \mathscr{C}$. In this inequality the right hand side does not depend on the specified $c \in \mathscr{C}$. Since, by definition of the set $\omega \subset (0,1)$, the points $x_i \in \omega$ are Lebesgue points of $b_{\nu i}$ as well as $g_{\nu i}$, $\nu=-m_0,\dots,n_0$, $i=1,\dots,l$, the statement is proved for $\mathscr{T}_{\nu 1}$.

Consider now $\mathscr{T}_{\nu 2}(\varepsilon, w)$, $\nu=-m_0,\dots,n_0$. Because of Lemma 4 there is a constant $K>0$ such that

$$\begin{aligned}&\varepsilon^{-1}|\mathscr{T}_{\nu 2}(\varepsilon, w)| \leqslant K(\|a_1^{\varepsilon c} - a_t^0\|_2 + \|a_2^{\varepsilon c} - a_s^0\|_2 + \|b_1^{\varepsilon c} - b_t^0\|_1 \\ &\quad + (\|b_2^{\varepsilon c} - b_s^0\|_1)\|z_\nu\|_0 + K(\|g_{\nu 1}^{\varepsilon c} - g_{\nu t}^0\|_1 + \|g_{\nu 2}^{\varepsilon c} - g_{\nu s}^0\|_1) \\ &\quad \forall \varepsilon \in (0, \delta_0)\,, \quad \forall c \in \mathscr{C}\,.\end{aligned}$$

Lemma 5 shows that the statement is also true for $\mathscr{T}_{\nu 2}$. Thereby the lemma is proved. ■

In the next section we have to deal with $\mathscr{T}_\nu(w^*)$, where $w^* \in \mathscr{X}$. However, for example, Lemma 7 holds only for $w \in \mathscr{X}_d$. This gap will be closed by the next lemma, which, roughly speaking, shows that $w \in \mathscr{X}_d$ can be chosen in such a way that $\mathscr{T}_\nu(w^*)$ and $\mathscr{T}_\nu(w)$ differ as little as we wish. In other words, our last lemma in this section gives a certain continuity property of $\mathscr{T}_\nu = \mathscr{T}_\nu(w)$.

LEMMA 9. *For any given $w^* = \{u_1,\dots,u_l,x_1^*,\dots,x_l^*,c_1^*,\dots,c_l^*\} \in \mathscr{X}$ and any given number $\eta>0$ there are measurable sets $\omega_i \subset (0,1)$, $i=1,\dots,l$, and a constant $\delta>0$ such that meas$(\omega_i \cap (x_i^*-\delta, x_i^*+\delta))>0$ and that the estimations*

$$|\mathscr{T}_\nu(w^*) - \mathscr{T}_\nu(w)| < \eta\,, \quad \nu=-m_0,\dots,n_0\,,$$

hold for each $w=\{u_1,\dots,u_l,x_1,\dots,x_l,c_1,\dots,c_l\} \in \mathscr{X}_d$ with

$$x_i \in \omega_i \cap (x_i^*-\delta, x_i^*+\delta)\,, \quad c_i \in \mathbb{R}_+ \cap (c_i^*-\delta, c_i^*+\delta)\,, \quad i=1,\dots,l\,.$$

Proof. Let $w^* = \{u_1,\dots,u_l,x_1^*,\dots,x_l^*,c_1^*,\dots,c_l^*\} \in \mathscr{X}$ be fixed and let $w \in \mathscr{X}_d$ be arbitrary, but of the form $w=\{u_1,\dots,u_l,x_1,\dots,x_l,c_1,\dots,c_l\} \in \mathscr{X}_d$. For the sake of brevity we put

$$d_{\nu i}(x) = b_{\nu i}(x) + g_{\nu i}(x)\,, \quad x \in (0,1)\,, \quad \nu=-m_0,\dots,n_0\,, \quad i=1,\dots,l\,,$$

where $b_{\nu i}$ and $g_{\nu i}$ are given in (13). Then we have

$$\mathcal{T}_\nu(w^*) - \mathcal{T}_\nu(w) = \sum_{i=1}^{l} (c_i^* - c_i) d_{\nu i}(x_i^*) + \sum_{i=1}^{l} c_i (d_{\nu i}(x_i^*) - d_{\nu i}(x_i)) ,$$
$$\forall x_i \in \omega , \quad \forall c_i \in \mathbb{R}_+ ,$$

and hence

$$|\mathcal{T}_\nu(w^*) - \mathcal{T}_\nu(w)| \le \sum_{\nu=-m_0}^{n_0} \sum_{i=1}^{l} |c_i^* - c_i||d_{\nu i}(x_i^*)|$$
$$+ \sum_{\nu=-m_0}^{n_0} \sum_{i=1}^{l} (c_i^* + |c_i - c_i^*|)|d_{\nu i}(x_i^*) - d_{\nu i}(x_i)| ,$$
$$\forall x_i \in \omega , \quad \forall c_i \in \mathbb{R}_+ .$$

Now let $\eta > 0$ be given. By definition of ω each $x_i^* \in \omega$ is a Lebesgue point of $d_{\nu i}$. Therefore, we may choose a constant $M_1 > 0$ such that

$$|d_{\nu i}(x_i^*)| < \frac{M_1}{l(m_0 + n_0 + 1)} , \quad \nu = -m_0, \dots, n_0 , \quad i = 1, \dots, l ,$$

and another constant $M_2 > 0$ with

$$c_i^* + \frac{\eta}{2M_1} < M_2 , \quad i = 1, \dots, l .$$

We choose also a positive $\delta_0 < \eta/2M_1$. Then, for any $c = \{c_1, \dots, c_l\} \in \mathbb{R}_+^l$ with $|c_i^* - c_i| < \delta_0$, $i = 1, \dots, l$, we get the estimations

$$|\mathcal{T}_\nu(w^*) - \mathcal{T}_\nu(w)| < \frac{\eta}{2} + M_2 \sum_{i=1}^{l} f_i(x_i) , \quad \nu = -m_0, \dots, n_0 , \tag{37}$$

where $x_i \in \omega$ is still arbitrary and where we have set

$$f_i(x) = \sum_{\nu=-m_0}^{n_0} |d_{\nu i}(x_i^*) - d_{\nu i}(x)| , \quad x \in \omega , \quad i = 1, \dots, l .$$

Obviously, for any $i = 1, \dots, l$ it holds $f_i \in L^1(0, 1)$, $f_i(x_i^*) = 0$ and the given $x_i^* \in \omega$ is a Lebesgue point for f_i. Therefore, f_i is at $x = x_i^*$ approximately continuous (see [11] Kap. XI, §6; [15] Sect. 4.4.), which means that there are measurable subsets $\omega_i \subset (0, 1)$ such that the Lebesgue density of ω_i at x_i^* equals one and that f_i is continuous at x_i^* relative to ω_i. In other words, for any $\delta > 0$ the sets $\omega_i \cap (x_i^* - \delta, x_i^* + \delta)$ have positive measures and for $\eta > 0$ given above there are such $\delta_i > 0$ that

$$0 \leq f_i(x) < \frac{\eta}{2M_2 l}, \quad x \in \omega_i \cap (x_i^* - \delta, x_i^* + \delta), \quad i = 1, \ldots, l.$$

Taking $\delta = \min\{\delta_0, \ldots, \delta_l\}$ from (37) the desired inequalities follow. ■

5. Proof of Theorem 1

As before $\{u_0, y_0\} \in L_m^\infty(0,1) \times H_0^1(0,1)$ denotes an optimal solution to our control problem **(P)**. If there are no integral constraints of type (3) at all, then Theorem 1 reduces to the necessary optimality condition

$$\left.\begin{array}{l} b(x, u, y_0(x), y_0'(x)) z_0(x) + g_0(x, u, y_0(x), y_0'(x)) \geq \\ b(x, u_0(x), y_0(x), y_0'(x)) z_0(x) + g_0(x, u_0(x), y_0(x), y_0'(x)) \\ \quad \forall u \in Q, \quad a.e.\, x \in (0,1), \end{array}\right\} \tag{38}$$

where $z_0 \in H_0^1(0,1)$ is the unique solution to (9) (for $\nu = 0$). This optimality condition can be easily verified. Indeed, since $\{u_0, y_0\} \in L_m^\infty(0,1) \times H_0^1(0,1)$ is a solution to **(P)** and because of Lemma 7, we have

$$0 \leq \varepsilon^{-1}(\mathcal{T}_0(u_{\varepsilon c}, y_{\varepsilon c}) - \mathcal{T}_0(u_0, y_0)) = \mathcal{T}_0(w) + \varepsilon^{-1}\mathcal{T}_0(\varepsilon, w),$$

for any $w \in \mathscr{X}_d$ and for all $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 = \varepsilon_0(w) > 0$ sufficiently small. From this, by Lemma 8, we get $0 \leq \mathcal{T}_0(w)\ \forall w \in \mathscr{X}_d$. Taking $w \in \mathscr{X}_d$ with $l = 1$ and $c_1 = 1$, this condition implies (38). (See also [10].)

Thus, in the following it is assumed that in (3) there is at least one integral condition to be satisfied. To prove now under this assumption Theorem 1 we introduce the notations

$$\begin{aligned} &Y = \{Y_{-m_0}, \ldots, Y_{-1}, Y_0, Y_1, \ldots, Y_{n_0}\} \in \mathbb{R}^{m_0+n_0+1}, \\ &Y' = \{Y_1, \ldots, Y_{n_0}\} \in \mathbb{R}^{n_0}, \\ &\mathcal{T}(w) = \{\mathcal{T}_{-m_0}(w), \ldots, \mathcal{T}_{n_0}(w)\}, \quad w \in \mathscr{X}, \end{aligned} \tag{39}$$

and define the sets

$$\mathscr{K}_0 = \{Y \in \mathbb{R}^{m_0+n_0+1}: Y_{-m_0} \leq 0, \ldots, Y_0 \leq 0, Y_1 = 0, \ldots, Y_{n_0} = 0\}, \tag{40}$$

$$\mathscr{K} = \{Y \in \mathbb{R}^{m_0+n_0+1}: Y = \mathcal{T}(w), w \in \mathscr{X}\}, \tag{41}$$

$$\mathscr{K}' = \{Y' \in \mathbb{R}^{n_0}: \exists Y_{-m_0}, \ldots, Y_0 \in \mathbb{R} \text{ such that } \{Y_{-m_0}, \ldots, Y_0, Y'\} \in \mathscr{K}\}. \tag{42}$$

Here $\mathcal{T}_\nu(w)$, $\nu = -m_0, \ldots, n_0$, is given in (35); in (39) and (42) it is assumed

$n_0 \geq 1$. Obviously, $\mathcal{K}_0$ is a convex cone with vertex at the origin. The same is true for the two other sets.

LEMMA 10. *$\mathcal{K} \subset \mathbb{R}^{m_0+n_0+1}$ and $\mathcal{K}' \subset \mathbb{R}^{n_0}$ are nonempty convex cones.*

Proof. It suffices to prove the statement for the set $\mathcal{K}$. If $\lambda > 0$ and $w = \{u_1, \ldots, u_l, x_1, \ldots, x_l, c_1, \ldots, c_l\} \in \mathcal{X}$, then with $w_\lambda = \{u_1, \ldots, u_l, x_1, \ldots, x_l, \lambda c_1, \ldots, \lambda c_l\} \in \mathcal{X}$ it holds $\mathcal{T}(w_\lambda) = \lambda \mathcal{T}(w)$ showing the cone property of $\mathcal{K}$. If $w^\mu = \{u_1^\mu, \ldots, u_{l_\mu}^\mu, x_1^\mu, \ldots, x_{l_\mu}^\mu, c_1^\mu, \ldots, c_{l_\mu}^\mu\} \in \mathcal{X}$, $\mu = 1, 2$, then with $w = \{u_1^1, \ldots, u_{l_2}^2, x_1^1, \ldots, x_{l_2}^2, c_1^1, \ldots, c_{l_2}^2\} \in \mathcal{X}$ we have $\mathcal{T}(w) = \mathcal{T}(w^1) + \mathcal{T}(w^2)$, from which the convexity of the cone $\mathcal{K}$ follows. ■

The evidence of introducing the cones $\mathcal{K}_0$ and $\mathcal{K}$ is given by the next lemma. This lemma will be essential in the subsequent discussion.

LEMMA 11. *If the cones $\mathcal{K}_0 \subset \mathbb{R}^{m_0+n_0+1}$ and $\mathcal{K} \subset \mathbb{R}^{m_0+n_0+1}$ defined in* (40) *and* (41), *respectively, can be separated, then Theorem* 1 *holds.*

Proof. By assumption, there exists a vector $\lambda = \{\lambda_{-m_0}, \ldots, \lambda_{n_0}\} \in \mathbb{R}^{m_0+n_0+1} \backslash \{0\}$ such that

$$(\lambda, Y) \leq 0 \quad \forall Y \in \mathcal{K}_0 \quad \text{and} \quad 0 \leq (\lambda, Y) \quad \forall Y \in \mathcal{K}\,.$$

The first condition yields $\lambda_{-m_0} \geq 0, \ldots, \lambda_0 \geq 0$ and the second one reads as

$$\sum_{\nu=-m_0}^{n_0} \lambda_\nu \mathcal{T}_\nu(w) \geq 0 \quad \forall w \in \mathcal{X}\,. \tag{43}$$

Since (43) implies the minimum condition (8), the lemma is already proved. ■

The further discussion is split up into two cases concerning the integral constraints (3). Namely:

(a) There is at least one equality condition.

(b) There is no equality condition at all.

In case **(a)**, which means $n_0 \geq 1$, we distinguish for the cone $\mathcal{K}' \subset \mathbb{R}^{n_0}$, defined in (42), the two subcases **(a1)** $\mathcal{K}' \neq \mathbb{R}^{n_0}$ and **(a2)** $\mathcal{K}' = \mathbb{R}^{n_0}$.

Let us begin by supposing **(a)** and **(a1)** are fulfilled. By ([3] Satz 2.25) the convex cone $\mathcal{K}'$ can be separated from the origin. That means, there is a vector $\lambda' = \{\lambda_1, \ldots, \lambda_{n_0}\} \in \mathbb{R}^{n_0} \backslash \{0\}$ with $0 \leq (\lambda', Y')$ $\forall Y' \in \mathcal{K}'$. Putting $\lambda = \{0, \ldots, 0, \lambda'\} \in \mathbb{R}^{m_0+n_0+1} \backslash \{0\}$ we again obtain (43), which also shows the validity of Theorem 1 in this case.

Now we consider the most difficult case characterized by **(a)** and **(a2)**. If we suppose that in this case Theorem 1 is not valid, then, by Lemma 11, the two cones $\mathcal{K}_0, \mathcal{K} \subset \mathbb{R}^{m_0+n_0+1}$ cannot be separated. We show this implies a contradiction to the optimality of $\{u_0, y_0\} \in L_m^\infty(0,1) \times H_0^1(0,1)$. More precisely, we construct an admissible pair $\{u_*, y_*\} \in L^\infty_m(0,1) \times H^1_0(0,1)$ to our control problem **(P)** such that $\mathcal{T}_0(u_*, y_*) < \mathcal{T}_0(u_0, y_0)$. This rather technical part of the proof will be done within several steps.

Step 1: Since $\mathcal{K}_0 \subset \mathbb{R}^{m_0+n_0+1}$ and $\mathcal{K} \subset \mathbb{R}^{m_0+n_0+1}$ cannot be separated, by ([12] Satz 9.1), it holds

$$\textit{rel int}\, \mathcal{K}_0 \cap \textit{rel int}\, \mathcal{K} \neq \emptyset .$$

Thus, there exists a vector

$$Y^* = \{Y^*_{-m_0}, \dots, Y^*_{n_0}\} \in \mathcal{K} \quad \text{with} \quad Y^*_{-m_0} < 0, \dots, Y^*_0 < 0 ,$$
$$Y^*_1 = \dots = Y^*_{n_0} = 0 .$$

Because of $\mathcal{K}' = \mathbb{R}^{n_0}$ we can choose an element

$$Y^0 = \{Y^0_{-m_0}, \dots, Y^0_{n_0}\} \in \mathcal{K} \quad \text{with} \quad Y^0_1 = \dots = Y^0_{n_0} = \kappa_1$$

and n_0 further elements

$$Y^\mu = \{Y^\mu_{-m_0}, \dots, Y^\mu_{n_0}\} \in \mathcal{K} \quad \text{with} \quad Y^\mu_\nu = \kappa_2 \delta_{\nu\mu} , \quad \nu, \mu = 1, \dots, n_0 .$$

Here $\kappa_1 < 0$ and $\kappa_2 > 0$ are two arbitrarily choosen constants and $\delta_{\nu\mu}$ denotes the Kronecker symbol. Convexity of $\mathcal{K}$ yields

$$(1-\lambda)Y^* + \lambda Y^\mu \in \mathcal{K} \quad \forall \lambda \in [0,1] , \quad \mu = 0, \dots, n_0 .$$

We take $\lambda \in [0,1]$ so small that

$$(1-\lambda)Y^*_\nu + \lambda Y^\mu_\nu < 0 , \quad \text{for all} \quad \mu = 0, \dots, n_0 , \quad \nu = -m_0, \dots, 0 \tag{44}$$

and introduce the constants $\kappa_3, \kappa_4 < 0$ and $\alpha, \beta > 0$ by setting

$$\left.\begin{aligned} \kappa_3 &= \min\{(1-\lambda)Y^*_\nu + \lambda Y^\mu_\nu \colon \mu = 0, \dots, n_0, \nu = -m_0, \dots, 0\} \\ \kappa_4 &= \max\{(1-\lambda)Y^*_\nu + \lambda Y^\mu_\nu \colon \mu = 0, \dots, n_0, \nu = -m_0, \dots, 0\} \end{aligned}\right\} \tag{45}$$

and

$$\alpha = \min\left\{\frac{2\kappa_1}{9\kappa_2\kappa_3}, -\frac{2}{3\kappa_3}\right\} , \quad \beta = \max\left\{\frac{6\kappa_1}{\kappa_2\kappa_4}, -\frac{2}{\kappa_4}\right\} , \tag{46}$$

respectively.

Step 2: By (41), there are elements

$$\bar{w}^\mu = \{u^\mu_1, \dots, u^\mu_{l_\mu}, \bar{x}^\mu_1, \dots, \bar{x}^\mu_{l_\mu}, \bar{c}^\mu_1, \dots, \bar{c}^\mu_{l_\mu}\} \in \mathcal{X} , \quad \mu = 0, \dots, n_0$$

such that

$$\mathcal{T}(\bar{w}^\mu) = (1-\lambda)Y^* + \lambda Y^\mu, \quad \mu = 0, \ldots, n_0. \tag{47}$$

Taking a constant η with

$$0 < \eta < \frac{1}{4} \min \left\{ \frac{\lambda \kappa_2}{\left[1 + \frac{\beta}{\alpha}(n_0+2)\right] n_0}, -2\lambda\kappa_1, -2\kappa_4 \right\}, \tag{48}$$

by Lemma 9, for each $\mu = 0, \ldots, n_0$ we find measurable sets $\omega_i^\mu \subset (0,1)$, $i = 1, \ldots, l_\mu$, and a positive number δ_μ such that $meas(\omega_i^\mu \cap (\bar{x}_i^\mu - \delta_\mu, \bar{x}_i^\mu + \delta_\mu)) > 0$ and

$$|\mathcal{T}_\nu(\bar{w}^\mu) - \mathcal{T}_\nu(w^\mu)| < \eta, \quad \nu = -m_0, \ldots, n_0, \tag{49}$$

holds for any

$$w^\mu = \{u_1^\mu, \ldots, u_{l_\mu}^\mu, x_1^\mu, \ldots, x_{l_\mu}^\mu, c_1^\mu, \ldots, c_{l_\mu}^\mu\} \in \mathcal{X}_d \tag{50}$$

with

$$x_i^\mu \in \omega_i^\mu \cap (\bar{x}_i^\mu - \delta_\mu, \bar{x}_i^\mu + \delta_\mu), \quad c_i^\mu \in \mathbb{R}_+ \cap (\bar{c}_i^\mu - \delta_\mu, \bar{c}_i^\mu + \delta_\mu),$$
$$i = 1, \ldots, l_\mu.$$

Again referring to Lemma 9 for each $\mu = 0, \ldots, n_0$ we can choose such elements $w^\mu \in \mathcal{X}_d$ of the form (50) and satisfying (49) that $x_i^\mu \neq x_j^\nu$ provided $i \neq j$ or $\mu \neq \nu$. This means for an arbitrary parametric vector $\zeta = \{\zeta_0, \ldots, \zeta_{n_0}\} \in \mathbb{R}_+^{n_0+1}$ the vector

$$w(\zeta) = \{u_1^0, \ldots, u_{l_0}^0, \ldots, u_1^{n_0}, \ldots, u_{l_{n_0}}^{n_0}, x_1^0, \ldots, x_{l_0}^0, \ldots, x_1^{n_0}, \ldots, x_{l_{n_0}}^{n_0},$$
$$\zeta_0 c_1^0, \ldots, \zeta_0 c_{l_0}^0, \ldots, \zeta_{n_0} c_1^{n_0}, \ldots, \zeta_{n_0} c_{l_{n_0}}^{n_0}\} \tag{51}$$

belongs to $\mathcal{X}_d$. Below $w(\zeta)$ will be considered in detail. For later purpose we note that, by (47) and (48), the inequalities (49) imply the following estimates for $\mathcal{T}_\nu(w^\mu)$:
If $\mu = 0$, then

$$\left.\begin{aligned} &\tfrac{3}{2}\kappa_3 < \mathcal{T}_\nu(w^0) < \tfrac{1}{2}\kappa_4, \ \nu = -m_0, \ldots, 0, \\ &\tfrac{3}{2}\lambda\kappa_1 < \mathcal{T}_\nu(w^0) < \tfrac{1}{2}\lambda\kappa_1, \ \nu = 1, \ldots, n_0, \end{aligned}\right\} \tag{52}$$

and if $\mu = 1, \ldots, n_0$, then

$$\left.\begin{array}{l}\frac{3}{2}\kappa_3 < \mathcal{T}_\nu(w^\mu) < \frac{1}{2}\kappa_4,\ \nu = -m_0, \ldots, 0\,, \\ -\eta < \mathcal{T}_\nu(w^\mu) < \eta,\ \nu = 1, \ldots, n_0,\ \nu \neq \mu\,, \\ \frac{1}{2}\lambda\kappa_2 < \mathcal{T}_\mu(w^\mu) < \frac{3}{2}\lambda\kappa_2\,. \end{array}\right\} \tag{53}$$

Step 3: For sake of brevity we put

$$a_\nu = \mathcal{T}_\nu(w^0)\,, \quad \nu = 0, \ldots, n_0\,, \quad b_\nu = \mathcal{T}_\nu(w^\nu)\,, \quad \nu = 1, \ldots, n_0\,, \tag{54}$$

and introduce the regular $(n_0+1) \times (n_0+1)$ matrix

$$\mathfrak{A} = \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & b_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_0} & 0 & \cdots & b_{n_0} \end{pmatrix}. \tag{55}$$

We choose two numbers $r > 0$ and $d < 0$ such that

$$\frac{r}{\alpha} < -d < \left(\sqrt{n_0+1} + \frac{1}{\sqrt{n_0+1}}\right)\frac{r}{\alpha}\,. \tag{56}$$

Let

$$\zeta^0 = -dy \quad \text{with} \quad y = \{y_0, y_1, \ldots, y_{n_0}\}\,, \quad y_0 = -\frac{1}{a_0}\,,$$

$$y_\nu = \frac{a_\nu}{a_0 b_\nu} \quad \text{for } \nu = 1, \ldots, n_0\,.$$

Then, for the closed ball $\bar{B}(\zeta^0, r)$ centered at ζ^0 and with radius r the inclusion $\bar{B}(\zeta^0, r) \subset int\, \mathbb{R}_+^{n_0+1}$ holds. Indeed, by (52) and (53), we have

$$\alpha \leqslant y_\nu\,, \quad \nu = 1, \ldots, n_0\,, \quad \text{and} \quad \|y\| \leqslant \beta\sqrt{n_0+1}\,. \tag{57}$$

Denoting by ζ_ν^0 and ζ_ν the νth component of $\zeta^0, \zeta \in \mathbb{R}^{n_0+1}$, respectively, for arbitrary $\zeta \in \bar{B}(\zeta^0, r)$ we have $|\zeta_\nu - \zeta_\nu^0| \leqslant \|\zeta - \zeta^0\| \leqslant r$, $\nu = 0, \ldots, n_0$ and, consequently,

$$\zeta_\nu \geqslant \zeta_\nu^0 - r = -dy_\nu - r \geqslant -d\alpha - r > 0\,, \quad \nu = 0, \ldots, n_0\,. \tag{58}$$

We take now an $\varepsilon > 0$ so small that

$$\varepsilon^{-1}\mathcal{T}_\nu(\varepsilon, w(\zeta)) < -\sum_{\mu=0}^{n_0} \zeta_\mu \mathcal{T}_\nu(w^\mu) \quad \forall \zeta \in \bar{B}(\zeta^0, r)\,,$$

$$\nu = -m_0, \ldots, -1\,, \tag{59}$$

$$\varepsilon^{-1}\mathcal{T}_\nu(\varepsilon, w(\zeta)) < \frac{r}{2\sqrt{n_0+1}\|\mathfrak{A}^{-1}\|} \quad \forall \zeta \in \bar{B}(\zeta^0, r),$$
$$\nu = 0, \ldots, n_0, \tag{60}$$

and that condition (14) with $c = c(\zeta) = \{\zeta_0 c_1^0, \ldots, \zeta_0 c_{l_0}^0, \ldots, \zeta_{n_0} c_1^{n_0}, \ldots, \zeta_{n_0} c_{l_{n_0}}^{n_0}\}$ is satisfied for any $\zeta \in \bar{B}(\zeta^0, r)$. Conditions (59) and (60) are justified by (52), (53), (58) and Lemma 8. The latter has the consequence that by way of $w(\zeta)$ as given in (51) a McShane-variation u_ζ of our optimal control u_0 is defined for arbitrary $\zeta \in \bar{B}(\zeta^0, r)$ (see (15)). By $y_\zeta = y(u_\zeta) \in H_0^1(0,1)$ we denote the related state. Because of Lemma 7, (35) and (51) we come to the relations

$$\begin{aligned}\mathcal{T}_\nu(u_\zeta, y_\zeta) &= \mathcal{T}_\nu(u_0, y_0) + \varepsilon\mathcal{T}_\nu(w(\zeta)) + \mathcal{T}_\nu(\varepsilon, w(\zeta)) \\ &= \mathcal{T}_\nu(u_0, y_0) + \varepsilon \sum_{\mu=0}^{n_0} \zeta_\mu \mathcal{T}_\nu(w^\mu) + \mathcal{T}_\nu(\varepsilon, w(\zeta)),\end{aligned}$$
$$\zeta \in \bar{B}(\zeta^0, r), \quad \nu = -m_0, \ldots, n_0. \tag{61}$$

Step 4: After having fixed $\varepsilon, r, -d > 0$, $\zeta^0 \in \mathbb{R}_+^{n_0+1}$ and $w(\zeta) \in \mathcal{X}_d$, we consider the nonlinear equation system

$$\left.\begin{aligned}&\zeta_0\mathcal{T}_0(w^0) + \varepsilon^{-1}\mathcal{T}_0(\varepsilon, w(\zeta)) = d, \\ &\sum_{\mu=0}^{n_0} \zeta_\mu \mathcal{T}_\nu(w^\mu) + \varepsilon^{-1}\mathcal{T}_\nu(\varepsilon, w(\zeta)) = 0, \quad \nu = 1, \ldots, n_0,\end{aligned}\right\} \tag{62}$$

for which we are going to verify that it has a solution $\zeta^* \in \bar{B}(\zeta^0, r)$. To this end, besides a_ν and b_ν introduced in (54), we define

$$b_\nu(\zeta) = \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{n_0} \zeta_\mu \mathcal{T}_\nu(w^\mu), \quad \zeta \in \bar{B}(\zeta^0, r), \quad \nu = 1, \ldots, n_0,$$
$$b_\nu(\varepsilon, \zeta) = \varepsilon^{-1}\mathcal{T}_\nu(\varepsilon, w(\zeta)), \quad \zeta \in \bar{B}(\zeta^0, r), \quad \nu = 0, \ldots, n_0,$$
$$\mathfrak{b}(\zeta) = \{0, b_1(\zeta), \ldots, b_{n_0}(\zeta)\}, \quad \zeta \in \bar{B}(\zeta^0, r),$$
$$\mathfrak{b}(\varepsilon, \zeta) = \{b_0(\varepsilon, \zeta), b_1(\varepsilon, \zeta), \ldots, b_{n_0}(\varepsilon, \zeta)\}, \quad \zeta \in \bar{B}(\zeta^0, r),$$

and $\mathfrak{d} = \{d, 0, \ldots, 0\}$. Then, using matrix $\mathfrak{A}$ defined in (55) system (62) reads as

$$\mathfrak{A}\zeta + \mathfrak{b}(\zeta) + \mathfrak{b}(\varepsilon, \zeta) = \mathfrak{d}, \quad \zeta \in \bar{B}(\zeta^0, r),$$

that means, (62) is equivalent to the fixed point equation

$$\mathfrak{f}(\zeta) = \zeta, \quad \zeta \in \bar{B}(\zeta^0, r),$$

where the vector valued function $\mathfrak{f} = \mathfrak{f}(\zeta)$ is given by

$$\mathfrak{f} = \mathfrak{A}^{-1}\mathfrak{d} - \mathfrak{A}^{-1}\mathfrak{b}(\zeta) - \mathfrak{A}^{-1}\mathfrak{b}(\varepsilon, \zeta), \quad \zeta \in \bar{B}(\zeta^0, r).$$

Because of

$$\mathfrak{A}^{-1}\mathfrak{d} = \zeta^0 \quad \text{and} \quad \mathfrak{A}^{-1}\mathfrak{b}(\zeta) = \left\{0, \frac{b_1(\zeta)}{b_1}, \ldots, \frac{b_{n_0}(\zeta)}{b_{n_0}}\right\}$$

we get

$$\|\mathfrak{f}(\zeta) - \zeta^0\| \leq \left(\sum_{\nu=1}^{n_0} \frac{b_\nu^2(\zeta)}{b_\nu^2}\right)^{1/2} + \|\mathfrak{A}^{-1}\|\left(\sum_{\nu=0}^{n_0} b_\nu^2(\varepsilon, \zeta)\right)^{1/2}, \quad \forall\zeta \in \bar{B}(\zeta^0, r).$$

Using (53) and (60) we see that

$$\frac{b_\nu^2(\zeta)}{b_\nu^2} = \frac{1}{\mathcal{T}_\nu^2(w^\nu)}\left(\sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{n_0} \zeta_\mu \mathcal{T}_\nu(w^\mu)\right)^2 \leq \|\zeta\|^2 \sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{n_0} \frac{\mathcal{T}_\nu^2(w^\mu)}{\mathcal{T}_\nu^2(w^\nu)}$$
$$\leq \|\zeta\|^2\left(\frac{2\eta}{\lambda\kappa_2}\right)^2 n_0, \quad \nu = 1, \ldots, n_0,$$

and

$$b_\nu^2(\varepsilon, \zeta) = \varepsilon^{-2}\mathcal{T}_\nu^2(\varepsilon, w(\zeta)) \leq \frac{r^2}{4(n_0+1)\|\mathfrak{A}^{-1}\|^2}, \quad \nu = 0, \ldots, n_0,$$

hold for any $\zeta \in \bar{B}(\zeta^0, r)$. Therefore, we obtain

$$\|\mathfrak{f}(\zeta) - \zeta^0\| \leq \frac{2\eta}{\lambda\kappa_2}\|\zeta\| n_0 + \frac{r}{2} \quad \forall\zeta \in \bar{B}(\zeta^0, r).$$

By (56) and (57), we have

$$\|\zeta\| \leq \|\zeta - \zeta^0\| + \|\zeta^0\| \leq r - d\|y\| \leq r\left[1 + \frac{\beta}{\alpha}(n_0 + 2)\right] \quad \forall\zeta \in \bar{B}(\zeta^0, r).$$

Taking eventually into account (48), we find $\mathfrak{f}$ maps the closed ball $\bar{B}(\zeta^0, r) \subset int\, \mathbb{R}_+^{n_0+1}$ into itself. Obviously, in virtue of Lemma 7 and formula (61), $\mathfrak{f}$ is continuous on $\bar{B}(\zeta^0, r)$. That is, by Brouwer's fixed point theorem (see, e.g. [14] §2.3), $\mathfrak{f}$ has a fixed point $\zeta^* \in \bar{B}(\zeta^0, r)$, which in accordance with our construction is also a solution to (62).

Step 5: In the previous step we have seen that the system (62) has a solution $\zeta^* = \{\zeta_0^*, \ldots, \zeta_{n_0}^*\} \in int\, \mathbb{R}_+^{n_0+1}$. Thus, via (51) we are given a $w_* = w(\zeta^*) \in \mathcal{Z}_d$ by means of which a McShane-variation $u_* = u_{\zeta^*} \in U_{ad}$ of u_0 is defined (see (15),

ε fixed); let $y_* = y(u_*) \in H_0^1(0,1)$ be the corresponding solution of state equation (2). Then from (61), (62) we get

$$\mathcal{T}_\nu(u_*, y_*) = \mathcal{T}_\nu(u_0, y_0) + \varepsilon \sum_{\mu=0}^{n_0} \zeta_\mu^* \mathcal{T}_\nu(w^\mu) + \mathcal{T}_\nu(\varepsilon, w_*) ,$$
$$\nu = -m_0, \ldots, -1 ,$$
$$\mathcal{T}_0(u_*, y_*) = \mathcal{T}_0(u_0, y_0) + \varepsilon \sum_{\mu=1}^{n_0} \zeta_\mu^* \mathcal{T}_0(w^\mu) + \varepsilon d ,$$
$$\mathcal{T}_\nu(u_*, y_*) = \mathcal{T}_\nu(u_0, y_0) , \quad \nu = 1, \ldots, n_0 ,$$

and from this, by (53) and (59),

$$\begin{aligned} &\mathcal{T}_\nu(u_*, y_*) < \mathcal{T}_\nu(u_0, y_0) , \quad \nu = -m_0, \ldots, -1 , \\ &\mathcal{T}_0(u_*, y_*) < \mathcal{T}_0(u_0, y_0) + \varepsilon d , \\ &\mathcal{T}_\nu(u_*, y_*) = \mathcal{T}_\nu(u_0, y_0) , \quad \nu = 1, \ldots, n_0 , \end{aligned} \tag{63}$$

showing that the pair $\{u_*, y_*\} \in L_m^\infty(0,1) \times H_0^1(0,1)$ satisfies the integral constraints (3). Since, furthermore, because of $\varepsilon d < 0$, inequality (63) is a contradiction to the optimality of $\{u_0, y_0\}$, thereby Theorem (1) is proved in case of **(a)** and **(a2)** are fulfilled.

Case **(b)** remains to be considered, that means the case that in (3) equality conditions do not occur. Since Lemma 11 is still valid (with $n_0 = 0$), we may in principle proceed as above. But, instead of having $n_0 + 1$ elements $w^\mu \in \mathcal{X}_d$ (cf. (50)) we have now only one element $w^0 = \{u_1, \ldots, u_l, x_1, \ldots, x_l, c_1, \ldots, c_l\} \in \mathcal{X}_d$ satisfying

$$\kappa_5 < \mathcal{T}_\nu(w^0) < \kappa_6 , \quad \nu = -m_0, \ldots, 0 ,$$

where $\kappa_5, \kappa_6 < 0$. We define $w(\zeta) = \{u_1, \ldots, u_l, x_1, \ldots, x_l, \zeta c_1, \ldots, \zeta c_l\} \in \mathcal{X}_d$, $\zeta \in \mathbb{R}_+$, (cf. (51)). Then we choose two numbers $r > 0$, $d < 0$ such that $d < \mathcal{T}_0(w^0) r$, define $\zeta^0 = d/\mathcal{T}_0(w^0)$ and take an $\varepsilon > 0$ so small that

$$\mathcal{T}_\nu(\varepsilon, w(\zeta)) < -\varepsilon\zeta\mathcal{T}_\nu(w^0) \quad \forall \zeta \in [\zeta^0 - r, \zeta^0 + r] \subset (0, \infty) ,$$
$$\nu = -m_0, \ldots, -1 ,$$
$$|\varepsilon^{-1}\mathcal{T}_0(\varepsilon, w(\zeta))| < r|\mathcal{T}_0(w^0)| \quad \forall \zeta \in [\zeta^0 - r, \zeta^0 + r] ;$$

(see Lemma 8, cf. (59), (60)). Arguing analogously as for (62) we can verify that the nonlinear equation

$$\zeta\mathcal{T}_0(w^0) + \varepsilon^{-1}\mathcal{T}_0(\varepsilon, w(\zeta)) = d , \quad \zeta \in [\zeta^0 - r, \zeta^0 + r] ,$$

has a solution $\zeta^* > 0$. We define $w_* = w(\zeta^*), u_* \in U_{ad}$, $y_* \in H_0^1(0,1)$ as before and come eventually to

$$\mathcal{T}_\nu(u_*, y_*) = \mathcal{T}_\nu(u_0, y_0) + \varepsilon\zeta^*\mathcal{T}_\nu(w^0) + \mathcal{T}_\nu(\varepsilon, w_*)\ , \quad \nu = -m_0, \ldots, -1\ ,$$
$$\mathcal{T}_0(u_*, y_*) = \mathcal{T}_0(u_0, y_0) + \varepsilon d\ ,$$

which again implies a contradiction to the optimality of $\{u_0, y_0\}$. Thus, Theorem 1 is proved also in the last one (**b**). ■

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